Exercise 72

If g is a twice differentiable function and $f(x) = xg(x^2)$, find f'' in terms of g, g', and g''.

Solution

Use the product rule and the chain rule to differentiate $f(x)$.

$$\begin{aligned}
f'(x) &= \frac{d}{dx}[f(x)] \\
&= \frac{d}{dx}[xg(x^2)] \\
&= \left[\frac{d}{dx}(x)\right] g(x^2) + x\left[\frac{d}{dx}g(x^2)\right] \\
&= (1)g(x^2) + x\left[g'(x^2)\cdot\frac{d}{dx}(x^2)\right] \\
&= g(x^2) + x\left[g'(x^2)\cdot(2x)\right] \\
&= g(x^2) + 2x^2g'(x^2)
\end{aligned}$$

Use them again to obtain $f''(x)$.

$$\begin{aligned}
f''(x) &= \frac{d}{dx}[f'(x)] \\
&= \frac{d}{dx}[g(x^2) + 2x^2g'(x^2)] \\
&= \frac{d}{dx}[g(x^2)] + 2\frac{d}{dx}[x^2g'(x^2)] \\
&= \left[g'(x^2)\cdot\frac{d}{dx}(x^2)\right] + 2\left\{\left[\frac{d}{dx}(x^2)\right]g'(x^2) + x^2\left[\frac{d}{dx}g'(x^2)\right]\right\} \\
&= [g'(x^2)\cdot(2x)] + 2\left\{(2x)g'(x^2) + x^2\left[g''(x^2)\cdot\frac{d}{dx}(x^2)\right]\right\} \\
&= 2xg'(x^2) + 2\{2xg'(x^2) + x^2[g''(x^2)\cdot(2x)]\} \\
&= 2xg'(x^2) + 4xg'(x^2) + 4x^3g''(x^2) \\
&= 6xg'(x^2) + 4x^3g''(x^2)
\end{aligned}$$